*Here is an excerpt from Chapter 7 LOGIC in the book Mathematical Milestones by Clement Falbo.*

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## Vacuously true statements

You will notice, in the $p$ *implies* $q$ table, that some strange looking results occur. For example, in the $p \Longrightarrow q$ column we claim that the statement is true when $p$ is false and $q$ is either true or false. That is "If $F$ then $T$" is a true statement and "If $F$ then $F$" is a true statement. What's up with that?

**Examples**

If the moon is made of green cheese, then apples are oranges. This is known as a "vacuously true" statement. It is true because the moon is not made of green cheese, so assuming that it is lets you draw any conclusion.

In some small room in my house which doesn't have any horses in it, I can make this statement: "If $h$ is a horse in this room, then $h$ has pink ears." This is a true statement because there are no horses in the room *which* do *not* have pink ears, since there is no horse in this room. The word *which* refers to the empty set, $\emptyset$ (sometimes called "phi") of such horses, because there are no such horses. The *which* has no referent. It is called the *irreferent which*.

Another example of the *irreferent which* is: *There is no integer* which *is greater than every other integer*.

Here is a poem that my wife wrote about this mathematical situation. When I told her about this concept she must have thought I said "an irreverent witch." This is what she wrote.

## The Irreferent Which (The Poem)

From: *Slices of Life*, a collection of poems by Jean Falbo.

THE IRREVERENT WITCH

An irascible old Witch
Entered my bedroom with a swish
Proclaiming
Every horse in this room has pink ears!

Whaaa? I sleepily bleated

There aren't any horses in here
Ha! She snerkled and snorted! Ha!
Vacuously true!
Vacuously true!
And she gave my love handles a poke
With a bony finger and an unmanicured nail, as she spoke

I rested my head on my pillow and I thought and I pensed
And I pensed and I thought
Only by logic can I banish this freak–and I was set to do it
But, inside I had an empty feeling I'd rue it
But, still I'd try to do it.
There, on my dresser, in my jewelry box is a horse of gold

With wings and mane and unicorn horn
That will prove you false, as sure as I'm born,
I shouted, as I leaped from my bed

That's it!
That's it!
You irreferent twit, I said

I'm outta here semper Phi
Null and void, she angrily retorted
And away she sped.

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This poem and the concept of *vacuously true* statements are based upon the Aristotelean Logic, requiring every premise to have either a value of True or False. But, in modern mathematics, there are some *non-Aristotelian* logics, ones in which there are more than two possible truth values.

## Non-Aristotelian, multi-valued logic

During the years from 1950 to 2000, mathematicians invented logical systems that had more than just the two truth values "True" and "False."

Some of these systems helped to develop modern applications in computer computations; for example, a *three-valued logic* can be implemented by having a switch that can be put into one of *three* different positions, closing and opening electrical circuits over three different routes. *Four-valued logic* has been used to increase the storage capacities of memory chips. Various applications have been found for other finitely many valued logics.

Others require that statements take on infinitely many values, say 1 for True, 0 for False and some fraction $x$ between 0 and 1 for any one of infinitely many different possibilities. These systems are useful in developing probability calculations. They help in making predictions in weather, medical research, business applications and political polls.

## Kleene logic

Here is an introduction to one of the three valued logics. It is interesting to compare the truth tables for this logic to those of Aristotelian logic. This was originally published by the American logician, Stephen C. Kleene in 1952. Let any premise have one of three different values, $T, U, F$, standing for *True*, *Unknown* and *False*. The *Not Table* is as follows

| $p$ | $\neg p$ |
|---|---|
| $T$ | $F$ |
| $U$ | $U$ |
| $F$ | $T$ |

Kleene's NOT

As can be seen, above, if a statement is not true it is false, if you negate an unknown, it is still unknown, the negative of a false statement is true. Now, here are the AND, $\wedge$, and OR, $\vee$, Tables in Kleene's logic.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $U$ | $U$ |
| $T$ | $F$ | $F$ |
| $U$ | $T$ | $U$ |
| $U$ | $U$ | $U$ |
| $U$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $U$ | $F$ |
| $F$ | $F$ | $F$ |

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $U$ | $T$ |
| $T$ | $F$ | $T$ |
| $U$ | $T$ | $T$ |
| $U$ | $U$ | $U$ |
| $U$ | $F$ | $U$ |
| $F$ | $T$ | $T$ |
| $F$ | $U$ | $U$ |
| $F$ | $F$ | $F$ |

Kleene's AND and OR Tables

In the AND, $\wedge$, table, $F$ and anything is $F$. While all three of: $U \wedge U$,

$U \wedge T$, and $T \wedge U$ are $U$. The only AND combination that produces $T$ is $T \wedge T$, just as with Aristotelian logic.

In the OR table, the only way an OR statement can be false is that both $p$ and $q$ must be false; in all other cases we get either $T$ or $U$.

In his logical system Kleene defines the implication (if then) statement to be the disjunctive negation.

$$p \Longrightarrow q, \text{ means } q \vee \neg p$$

If you set up the truth table for $p \Longrightarrow q$ you will see that it is compatible with Aristotelian logic. This shows that Kleene's logic is actually a generalization of Aristotelian logic, because if you omit the $U$ possibility in all tables, you get Aristotelian. An interesting exercise is to state the denial of $p \wedge q$ in Aristotelian and compare it to the denial of $p \wedge q$ in Kleenean logic.

Hint: In regular (Aristotelian) logic, you can write the following theorem:

$$\neg(p \wedge q) = \neg p \vee \neg q$$

Have fun comparing this to $\neg(p \wedge q)$ in Kleene's logic, by using the Kleene tables for *and*, *or*, and *not*.

## From logic to axioms

Aristotle's logic of 330 BCE blossomed into Euclid's geometry of 280 BCE, when Euclid published his 13 volume work, *Elements of Geometry*. What Euclid did was to organize all known geometry into a coherent collection of definitions, axioms and theorems and showed that logic could be used to prove the theorems. This began the idea that a mathematical system could be arranged in such a way that you could derive true statements in an axiomatic system.