Here are some pages fom Chapter 9, Matrix Applications, of my new book: Mathematical Milestones. Clement Falbo.

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A program called *DeepStack*, written in 2017 seems to be able to eke out wins against the world's best Texas hold'em poker players. This program builds in bluffing buffers as well as other Artificial Intelligence algorithms. It learned how to handle bluffing by playing thousands of games against thousand of players with different simulated bluffing personalities.

1 Matrix games

To understand how game theory got started before it evolved into the sophisticated topic we know as Artificial Intelligence, we will give examples of two-player (or two-team) games in which each side has a restricted number of choices. These will be perfect information games because the available choices for each side are clearly displayed. Each side will be able to calculate their own and their opponents possible gains and losses depending upon the choices each side makes. Each side reaps a reward or suffers a loss according to a mutually agreed upon prearranged scheme. If one side wins exactly what the other side loses, then it is a *zero sum* game, otherwise a *non-zero sum* game.

The word "game" is used here as a catch-all term. It can be used to actually represent a conflict between two parties, competing for some dominance, or for some prize. It could also, be a negotiation, in which each side is attempting to get the best possible deal. It could simply be looking at a list of options and a list of their concomitant consequences suggesting or even dictating some action on the given options. It was probably in this sense of decision making that, in 1947, John von Neumann, a physicist, and Oskar Morgenstern, an economist, wrote their book *Theory of Games and Economic Behavior* in which they developed the basic assumptions for "Game Theory."

The most fundamental ones are:

1. The desire to win postulate, and
2. The Intelligent player postulate.

We can easily compare how the play of real games and the play of theory games use these two postulates.

Real games vs Theory games

- *Clarity of goal–the desire to win postulate*

In real games, players may have fuzzy, unclear, or even contradictory, motivation to win a "unit of utility." They may wish to "win by losing." In playing a weaker opponent or a child, the stronger players may let the weaker player win; or they may not try to take every point. This could happen in a teaching situation.

In Theory games, the players always try to optimize their gain.

- *Knowledge of available strategies–the intelligent player postulate*

In real games, players may not be mindful of the opponents' possible moves, or they may misjudge the opponents, assuming that they will not make the best move because they are not playing intelligently.

In theory games players are aware of each others' possible strategies, and assume that their opponents are intelligent players.

1.1 A game as a decision-making tool.

Game theory may be applied to a decision making problem, in which your opponent is not a person or some other contentious entity trying to outwit you. It could be the weather, or the future, or some set of possible random events you must anticipate in order to choose your best possible plan of action. You may not have perfect information, but you could have a good idea about the most serious and most probable challenges you will be facing.

Example:

Let's say that a high school student is trying to decide among three colleges that have invited her to apply. She sets up a matrix game in which the row "player" is her decision to select one of the three colleges, and the column player is the future, that is, a set of contingencies that determine her future job interests, based upon her evolving, self-assessed inclinations and aptitudes.

She partitions the three colleges into

r_1 : Technological Institute
r_2 : Business College
r_3 : Liberal Arts College

The student's future job interests, the column player, is partitioned into

c_1 : Engineer
c_2 : Mathematician
c_3 : Musician
c_4 : Business Manager

She constructs Table 9.10 in her attempt to make a decision. This is a typical matrix game; the table, itself, is called a *payoff matrix;* the numbers in the table are called utilities (or values). When a row is selected, and a column is selected then the payoff is the utility in the intersection of that row and that column.

Column Player

Row Player

	c_1	c_2	c_3	c_4
r_1	14	6	2	4
r_2	4	3	1	12
r_3	10	8	9	9

Table 9.10 Matrix Game

If she selects r_1, the Tech Institute, *and* her future career choice is engineering, c_1, row 1, column 1, (r_1, c_1), then the utility is high, 14. But, if her future choice is mathematics, c_2, then the Tech Institute gives her a lower utility, 6. If she ends up in music, c_3, then the Tech Institute gives her an even lower payoff, 2, found in (r_1, c_3). Finally, the Tech Institute, Business Career, (r_1, c_4) pay off is 4.

Problem:

1. Suppose the high school student selects r_2, show that her best career choice would be as a business manager.

2. If she chooses r_3 show that her best future career choice is to become an engineer.

3. If her career leads toward becoming a mathematician, what is her best college choice?

4. What is the smallest number in each row and what does it mean for future careers?

5. Is there a number that is the smallest in its row and the largest in its column?

Answers:

1. Look at r_2, then going across that entire row, looking at the numbers column by column and you find the greatest utility for this row is in c_4, business manager.

2. For r_3, the largest value is in c_1, engineer.

3. In this case we look at the column c_2 and try to find the college that gives her the largest utility, and that is r_3.

4. The smallest value is any row is the least amount of utility you could get from that college for the career defined in that column.

5. Yes, the number 8 in Row 3, Column 2 is a number that is both its row minimum and its column maximum.

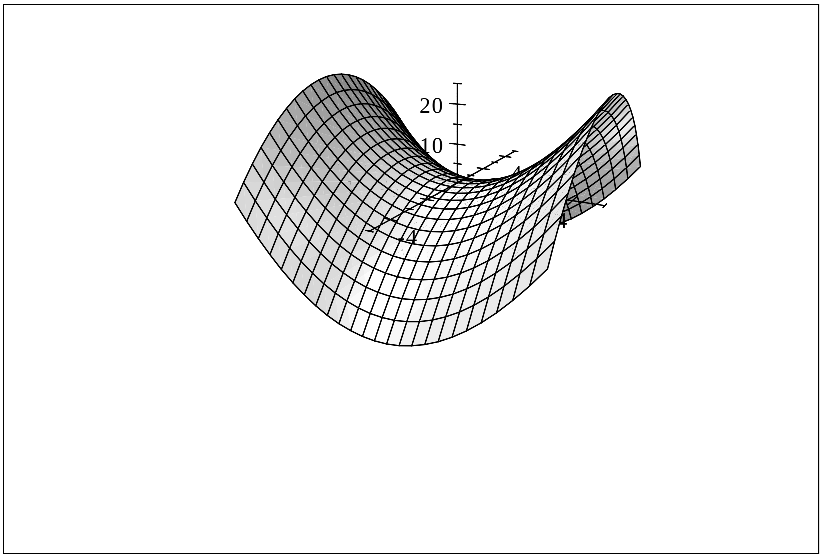

Figure 9.4 A SADDLE POINT = MINIMAX STRATEGY

Under the uncertain conditions about a future career, the row r_3 is the *safest* choice in terms of career and college. There are future careers in which some other college has a greater reward, but *all* colleges, other than r_3, have some future career with a smaller reward.

The selection of a number that is the minimum in its row and the maximum in its column is called the *minimax strategy*. The number in that position is called a "saddle point" for the game. See Figure 9.4. It depicts a surface that has a point that is simultaneously the minimum in one direction and the maximum in the perpendicular direction. It is the best that can be done for that matrix.

Table 9.11, is another example of a saddle point matrix game.

Column Player

		c_1	c_2	c_3	c_4
Row Player	r_1	R wins 1	C wins 3	C wins 1	C wins 2
	r_2	R wins 5	R wins 10	C wins 2	C wins 4
	r_3	R wins 7	R wins 5	R wins 9	C wins 1

Table 9.11

This game is unlike the college selection game in which the column player was a fixed entity, the "future career." In that game the column player did not deliberately switch around selecting various columns hoping to trip up the row player. The game in Table 9.11 has two aggressive active players. Let's call them the Raven and the Crow, respectively, for Row and Column. In order to maintain gender neutrality in our discussion, we will refer to a team on either side as *it*.

Let's imagine that the following takes place. Raven (R) and Crow (C) walk into the room. They sit down and are shown the payoff table for the first time. R's task is to *secretly* pick a row to play; the whole row is R's pick. Now, C's task is to *secretly* select a column to play, again, the whole column. Neither player knows its opponent's choice. They simultaneously reveal their choices and are rewarded or penalized according to what is in the intersection of this row-column combination.

Although neither player knows what the other will play, they can easily figure out what the other player *should* play because they both can see the entire payoff matrix.

Problems:

1. Show that there is a column that C can play and never lose.
2. Show that R does not have a no-lose row
3. When the game is to be played over and over again, what row is safest for R?
4. Why should C never play c_1, nor c_2, nor c_3?

Solutions:

1. If C selects column c_4, it will never lose no matter what R plays.
2. All three of the rows have at least one loss to C, so R does not have a no-lose row.
3. R, being an intelligent player, knows that C will play its no lose column c_4, so R must minimize its losses by *always* playing r_3. This is its safest row. Suppose R thinks about playing some other row, say, r_2, hoping C is going to play c_2 giving R a large 10 point win, or column c_1 giving R 5 points, but this is an unintelligent play by R because C, by continuing play to c_4 makes R suffer a 4 point loss. Similarly, R should not think about playing r_1.
4. Because C is intelligent and knows that R will always play r_3, making C lose 7, 5, or 9 points, if it tried to play c_1, c_2, or c_3.

It should be clear that the matrix game in Table 9.11 is biased in favor of the C player. It turns out to also be a game which has a saddle point; we will rewrite it in a standard matrix form, that lets us recognize such games and immediately apply a special strategy for solving them.

The payoff matrix in the previous game can be expressed as follows.

Crow

		c_1	c_2	c_3	c_4
Raven	r_1	+1	−3	−1	−2
	r_2	+5	+10	−2	−4
	r_3	+7	+5	+9	−1

This is the same matrix as the one in Table 9.11, written entirely from Raven's point of view. The *pluses* mean Raven *wins* and the *minuses* mean Raven *loses* (Crow wins). So, for example the +5 in the second row, first column means that R wins 5 (and C loses 5), while the −4 in the second row, fourth column means R loses 4 and C wins 4.

The boxed in number $\boxed{-1}$ in the third row fourth column is the saddle point. It is a number that is the smallest in its row and largest in its column. This means that when R selects r_3, then 1 is the amount that R loses if C plays correctly. This is the best R can do, in this biased game. Also, when C plays column c_4, 1 point is the maximum that C can win if R plays correctly.

1.2 Saddle point strategy

As we mentioned above, the saddle point strategy is also called the minimax strategy. Not all games have a saddle point, but in any that do, this is the best strategy for both players, any other attempts to select a different row by R will be worst for R. Similarly, if C tries to select a column other that the one containing the saddle point, then C will fare worse. If a game has more than one saddle point, playing for any one of them is also the best each player can do. When there is no saddle point, we will play for a mixed strategy.

1.3 Mixed Strategy

What is the best way to play a game that does not have a saddle point? This question has a practical answer only if one anticipates playing the game several times. This is because the best strategy has the players switching back and forth randomly among the rows and columns. This strategy requires the players to have previously calculated the probabilities with which they must select their rows and columns to insure a beneficial outcome independent of the opponents choices.

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After a few more examples, I discuss the all strategies used in all n by 2, zero-sum games. This chapter and introduce the important topic of non-zero sum games. Flow charts, such as tthe following example are used to summarize the different strategies.

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1.4 Optimal strategies for all two by two games

The formulas for two-by-two games can be depicted in the following flowchart.

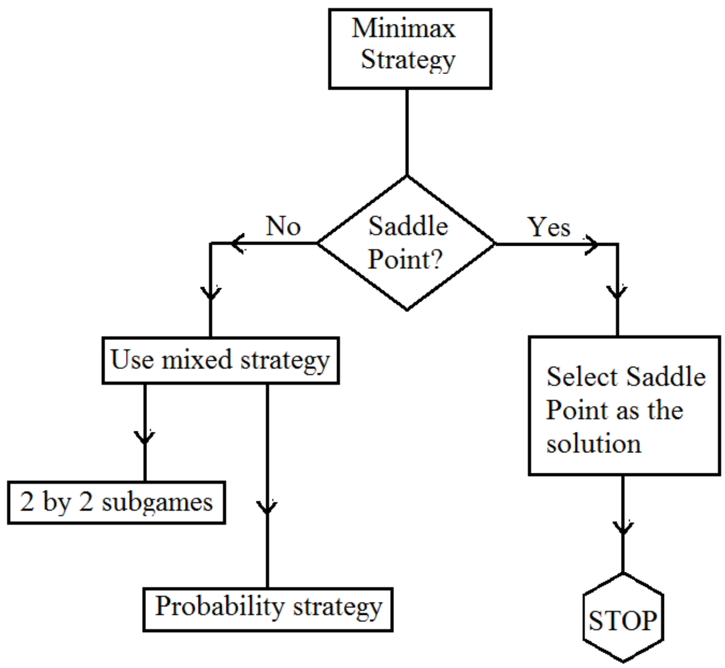

FIGURE 9.5 GAME STRATEGY

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Finally, as we depart this chapter, we move on the Chapter 10, where we learn about the spectacular applications of modern mathematics that was spawned by game theory.

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1.4.1 Operations research

In the next chapter we see a mathematical concept called *Operations Research* or *Linear Programming*. This concept is intimately related to game theory. It is widely used in business, hospitals, military, communication, and in any other field in which it is important to manage large networks to optimize performance, subject to constraints. This could mean maximizing profits, minimizing costs while simultaneously conserving resources and satisfying demand. It is undoubtedly the most, other than calculus, wide spread "use" of mathematics in the anthropocene age.