Here is an excerpt from Chapter 2 BREAKTHROUGHS from my book Mathematical Milestones

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1 How to understand abstract notation

When we start solving problems in various abstract mathematical systems we must pay attention to the currency of any definition or notation specific to the particular system under discussion. For example, in a set of matrices, a mathematician might say let A and B be any two n by n matrices, and define $A + B$ as an n by n matrix obtained by adding corresponding elements. But later on, in a different problem this same mathematician might say, let A and B be two circles and define $A + B$ to be a circle whose area is the sum of the areas of A and B. This means just concentrate on the current meaning of + not the meaning previously assigned in a different context

This is the nature of *fresh definitions* when we say "define your terms," used in philosophy, mathematics and legal contracts. The words in any specific contract mean only what they were defined to mean in that contract, not what they may have meant in some other unrelated one.

1.1 What properties can binary operations have?

1.1.1 Closure Law

Here, we will be talking about abstract, undefined, generalized combinations of two things. The actual nature of the combination should emerge, or at least settle down to a comprehensible concept as we say more and more about it. At the outset, however, we will say two things: First that we are starting with a well defined *set* of identifiable elements, and second that a combination of elements can qualify as a binary operation on that set only if it satisfies the *closure* law, meaning that the *result* of the combination must be one of the elements in the original given set. Instead of using just the notations + or $\cdot$ or $\times$, we will subject you to an abstract operator, $\circledast$,and define what it stands for in different settings.

1.1.2 Associative Law

A law that may be satisfied by the addition of matrices and other types of things is $A + (B + C) = (A + B) + C$. This is called the "associative law." It lets us conveniently "group" elements that are being added together. You may remember this from your arithmetic class in elementary school.

$$\begin{aligned} 3 + 7 + 5 + 4 &= (3 + 7) + (5 + 4) \\ &= 10 + 9 = 19 \end{aligned}$$

The associative law is important in problems in which you need to combine (by adding or multiplying) more than just two numbers at a time. If we tried to apply the associative law to adding odd numbers it would make no sense, because neither $B+C$, nor $A+B$ are numbers in the system of odds. (You might say, "they are not odd enough to belong to the odd fellows.")

In general, for any operator $\circledast$ the associative law would be: $x \circledast (y \circledast z) = (x \circledast y) \circledast z$. A binary operator without the associative law works only with two elements at a time. In such a case, we would not be able to infallibly determine the value of expressions such as $x \circledast y \circledast z$. This would severely limit the usefulness of the mathematical system with that operator.

1.1.3 Commutative Law

The commutative law: $x \circledast y = y \circledast x$, for example, $A+B = B+A$ says that if you add any two elements of a set you get the same thing regardless of the *order* in which you add them. There are examples in which the commutative law does not hold, for example as we learned in Chapter 2, matrix multiplication is not commutative.

Also, any operator, $\circledast$, is not commutative if it requires one process be completed before the other is begun and is such that these processes could not be reversed. For example, in a bottling plant, if you fill the bottle and then cap the bottle, you are OK, but it would be disastrous if you set up the machinery to cap bottles then tried to fill them.

1.1.4 Identity Law, neutral elements

Let $\circledast$ be a closed binary operator on a set S of things $\{x, y, ...\}$. If there is some element e (not to be confused with the famous mathematical constant $e = 2.71828...$, nor the famous physical constant $e = mc^2$) in S such that, for all x in S, $x \circledast e = x$, then the operator, $\circledast$ satisfies the *identity law* and the element e is called the *identity (or neutral) element* for $\circledast$ in S.

Examples: In ordinary arithmetic if the operator $\circledast$ is addition, $+$, then the identity element e is 0 because $x + 0 = x$. If, on the other hand, the operator $\circledast$ is multiplication, $\times$, then the identity element is 1 because $x \times 1 = x$. In the set $\mathcal{O}$ of odd numbers, the number 1 is the identity element for multiplication because 1 is odd and if x is any odd then $x \times 1 = x$. And, $1 \times x = x$, since multiplication is commutative in the integers.

1.1.5 Inverse Law

If S is a system that has a closed operator $\circledast$, and e is the neutral element for $\circledast$, and if for each element x in S, there is an element y in S such that $x \circledast y = e$, then y is called the *inverse* of x for the operator $\circledast$. In ordinary arithmetic, if the operator $\circledast$ is $+$ where the identity element is 0, and if x is any element in, S, then the inverse of x is denoted by $-x$ thus, $x + (-x) = 0$. Likewise, if the operator $\circledast$ is multiplication, $\times$, where the identity e is 1, and if x is in S and

$x \neq 0$, then the inverse of x is denoted by x^{-1} or by $1/x$. That is, $x \times x^{-1} = 1$, or $x \times (1/x) = 1$. Notice, here, that since we have required that $x \neq 0$, we are saying that 0 has no multiplicative inverse. This brings up the question: can the two neutral elements (the one for addition and the one for multiplication) be the same? Is it possible that $1 = 0$? The answer is no unless $+$ and $\times$ are both the very same operator. Unfortunately, the reader must keep track of which operator, $+$ or $\times$ is being discussed as to whether the identity is 0 or 1 and which inverse is being discussed as to whether it is $-x$ or x^{-1}. In this regard, special care needs to be taken when reading about the identities and inverses for a general operator such as $\circledast$, in which the neutral elements may be neither 0 nor 1. This is related to the admonition to "define your terms" and keep track of your definitions mentioned above.

2 Addition

2.1 What is addition?

Before the invention (or discovery?) of group theory, we had a pretty good idea of what addition meant. Consider two line segments of lengths a and b joined at one end point and placed along the number line, we get a line segment of $a + b$. Or, if we have a jar of marbles and drop another marble into the jar we have added it. But today, we can think of adding an element a and an element b of that set by combining them somehow and precisely describing the rule by which they are combined. In other words, it takes some *defining statement* of the relationship we assign to the resulting outcome that we are willing to call $a + b$.

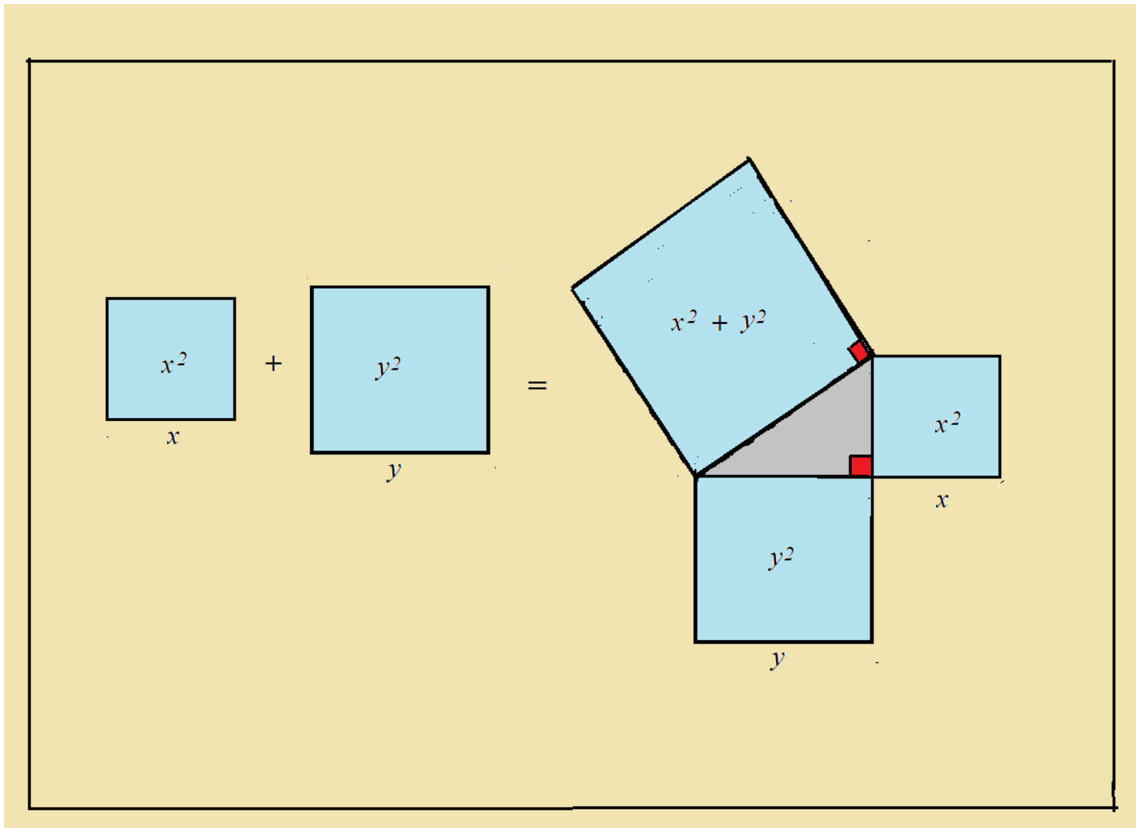
2.2 Adding Geometric Figures

In geometry, if we concoct a definition for adding two squares we want the sum to be another square and not a triangle or a circle, or some other geometric figure. As we have previously said, this restriction is called the *closure law* for addition, meaning that the system is "closed" in the sense that adding two elements of a given collection will **not** produce a new object *outside* of that collection.

Example: *Adding Squares.*

Let A and B be two squares whose areas are x^2 and y^2, respectively. We define addition of these two squares to be *a square* whose area is $x^2 + y^2$.

As you can see in Figure 3.1 such a new square can be created by making the sides x and y be the perpendicular sides of a right triangle, then the result (by the Pythagorean Theorem) is the square on the hypotenuse, whose area is $x^2 + y^2$.



Adding squares by adding their areas

Using this definition, we can say that the addition is closed, associative, and commutative